Exercise 7.2.14

The rate of evaporation from a particular spherical drop of liquid (constant density) is proportional to its surface area. Assuming this to be the sole mechanism of mass loss, find the radius of the drop as a function of time.

Solution

Let dm/dt be the rate that mass changes with respect to time, and let S be the sphere's surface area. Then the following relationship can be written from the first sentence.

$$\frac{dm}{dt} \propto -S$$

The symbol $\propto$ means "proportional to," and the minus sign is included because an increase in S leads to a decrease in dm/dt. This proportionality can be changed to an equation by including a proportionality constant k. This constant is different for different liquids and measures how fast a given liquid evaporates.

$$\frac{dm}{dt} = -kS$$

Mass is the product of densiy ρ and volume V.

$$\frac{d(\rho V)}{dt} = -kS$$

Since the density is assumed to be constant, it can be pulled in front of the derivative.

$$\rho \frac{dV}{dt} = -kS$$

For a sphere, $V = (4/3)\pi r^3$ and $S = 4\pi r^2$.

$$\rho \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k(4\pi r^2)$$

$$\rho\left(4\pi r^2 \frac{dr}{dt}\right) = -k(4\pi r^2)$$

Divide both sides by $(4\pi r^2)\rho$ to solve for dr/dt.

$$\frac{dr}{dt} = -\frac{k}{\rho}$$

Integrate both sides with respect to t.

$$r(t) = -\frac{kt}{\rho} + C$$

Assume that at $t = 0$ the sphere radius is R_0 so that the initial condition is $r(0) = R_0$. Apply it now to determine C.

$$r(0) = C = R_0$$

Therefore,

$$r(t) = R_0 - \frac{kt}{\rho}.$$